MAE 290B. Homework 3

Winter 2018

Due Thursday, Feb. 22, in class

Note: *You are expected to provide legible and clear solutions for the homework. Otherwise points will be deducted. Plots and sketches must be labeled and the axis well defined. Staple the document and number the pages. Torn-out pages are not allowed.*

1. The linear advection-diffusion equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}\,, \tag{1}$$

with u and α as positive constants, is to be numerically solved. A RKW3 (*Numerical Renaissance. Section 10.4.1.3*) scheme is to be used for the time integration with a second-order, central approximation for the spatial derivatives.

a) Perform an analysis of the spatial discretization to obtain the modified wavenumber/s k' as a function of the wavenumber k.

b) Obtain the amplification factor of the numerical method using Fourier modes with modified wave number/s k'.

c) Perform a stability analysis of the method. What is the restriction on the CFL number for the numerical method to be stable when $\alpha = 0$? What is the restriction on the diffusion number for the numerical method to be stable when $u = 0$? Can you come up with a simple combined restriction for arbitrary values of u and α?

d) Eq.(1) is going to be numerically integrated in a domain $0 \leq x \leq 1$. The initial condition is

$$T(x,0) = \begin{cases} 0 & x \leq 0 \\ \sin(\pi x)\left(\cos(5\pi x) + \sin(20\pi x)\right) & 0 \leq x \leq 1 \\ 0 & 1 \leq x\,. \end{cases} \tag{2}$$

i. Advection: The parameters are $u = 0.5$, $\alpha = 0$. The boundary condition is $T(x = 0, t) = 0$. Solve for $T(x,t)$ over the time interval $0 \leq t \leq 2$. Explain how would you choose Δx and Δt based on accuracy and stability. Compare the numerical solution with the exact one at $t = 0, 0.5, 1, 2$ s and discuss the evolution of $T(x,t)$. In order to avoid numerical reflections use first-order backward finite difference in the right boundary.

ii. Diffusion: The parameters are $u = 0$, $\alpha = 0.01$. The boundary conditions are $T(x = 0, t) = 0$ and $T(x = 1, t) = 1$. Solve for $T(x,t)$ over the time interval $0 \leq t \leq 10$. Explain how would you choose Δx and Δt based on accuracy and stability. Plot and discuss the evolution of

$T(x,t)$ at $t = 0, 0.025, 0.1, 0.5, 1, 10$ s. Is the characteristic diffusion time equal for all the wavenumbers?

iii. The diffusion equation with variable diffusivity $\alpha(x)$ is

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial T}{\partial x}\right), \tag{3}$$

the diffusion coefficient is

$$\alpha(x) = \begin{cases} 0.001 & x \leq 0.5 \\ 0.01 & 0.5 \leq x\,. \end{cases} \tag{4}$$

The boundary conditions are $T(x = 0, t) = 0$ and $T(x = 1, t) = 1$. Solve for $T(x,t)$ over the time interval $0 \leq t \leq 10$. Plot and discuss the evolution of $T(x,t)$ at $t = 0, 0.025, 0.1, 0.5, 1, 10$ s. Could you give an example where this behaviour can be observed?

2. The inviscid Burger's equation,

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = 0$$

with the following initial condition

$$u(x,0) = \begin{cases} e^{-\frac{(x-1)^2}{0.18}} & x \leq 2 \\ 0 & 2 \leq x\,. \end{cases} \tag{5}$$

is to be solved in the domain $0 \leq x \leq 5$ and $0 \leq t \leq 10$. The boundary condition is $T(x = 0, t) = 0$.

a) Use the structure of the code in Problem 1 to obtain the solution at $t = 0, 0.5, 1, 1.5, 5, 10$. Use a second-order, central approximation for the spatial derivatives.

b) Repeat part 2.a) with upwind first-order FD approximation for the spatial derivatives.

c) Discuss numerical inaccuracies in your solutions from part a) and b). What alternative methods would you suggest to solve this nonlinear equation? Justify your answer.