MAE 290B. Homework 1

Winter 2018

Due Thursday, Jan. 25, in class

1. A function $f(x)$ is known at equispaced points x_j, $j = 1, 2, \dots, N$ with spacing h.

a) Obtain a central approximation to $f'(x_j)$ that involves no more than three points $j-1, j, j+1$. What is its accuracy?

b) Obtain a Padé approximation for $f'(x_j)$ that involves three points $j-1, j, j+1$ and is fourth-order accurate.

c) Consider the function e^{ikx}. Compare the accuracy of both methods using modified wavenumber analysis. Plot the non-dimensional modified wavenumber $k'h$ as a function of kh.

30 points

2. The following IVP is to be numerically integrated to obtain y at $t = 1000$ s:

$$y' = \left(-\frac{1}{\tau} + i\omega\right) y \quad ; \quad y(0) = 1\,. \tag{1}$$

Use the θ method for solving $y' = f(y)$ as follows:

$$y_{n+1} = y_n + h[\theta f_{n+1} + (1-\theta) f_n]\,, \tag{2}$$

where $\theta \in [0, 1]$ is a fixed parameter.

a) Using the model problem $y' = \lambda y$ perform a stability analysis for $\theta = \{0, \frac{1}{2}, 1\}$. How does the choice of θ affect the stability? Sketch the stability regions for the three cases. Recall that we are assuming that $\lambda_R \leq 0$ i.e the exact solution is bounded.

b) Write a computer program that implements the θ method. Use it to obtain the solution for the initial times $0 \leq t \leq 60$ and the later times $940 \leq t \leq 1000$ with a user-prescribed θ, ω and h. Let $\omega = 0.25$ s^{-1}, $\tau = 800$ s and consider two cases: $\theta = 1/4$ and $\theta = 3/4$. Vary the time step $h = 0.01, 0.1, 1$ s and discuss the influence on the solution. Compare the two methods to the exact solution in terms of accuracy, stability and convergence.

c) Assume the characteristic decay time $\tau \to \infty$ so that Eq. (1) simplifies to

$$y' = i\omega y \quad ; \quad y(0) = 1\,.$$

Analytically obtain the amplitude and phase error of the method with $\theta = 3/4$ and time step h. Simplify the error expressions assuming that $\omega h << 1$.

40 points

3. A fourth-order Runge-Kutta scheme (RK4) is used to integrate the model linear problem:

$$y' = \lambda y \quad ; \quad y(0) = 1\,. \tag{3}$$

a) Obtain the stability restriction on the time step h for $\lambda \in \mathbb{C}$. Show the solution as a stability diagram. What is the restriction on h when $\lambda \in \mathbb{R}$?

b) Write a program that integrates Eq.(1) using a RK4 scheme (P. Moin *Engineering Numerical Analysis* Eq.(4.30)). Vary the time step $h = 0.01, 0.1, 1\ \mathrm{s}$ and compare the results with the solutions of problem 2b.

30 points