MAE 214A, Spring 2017
Homework 1
Due Thursday, April 20, in class

Guidelines: Please turn in a *neat* homework that gives all the formulae that you have used as well as details that are required to understand your solution. Note that the work must be your own work (no use of solutions, collaboration with other students or other sources).

1. Problem 2.10, Pope: derive the transport equation for enstrophy, $\omega_i\omega_i$, the squared magnitude of the vorticity vector. Start from the transport equation for velocity, u_i, assuming incompressible flow. What is the physical significance of each term in the enstrophy equation?

2. Problem 4.3, Pope.

3. Consider the Rayleigh-Bénard problem. Two infinite flat plates are separated vertically by a distance d. The lower plate with temperature T_l is hotter than the upper plate with temperature $T_u = T_l - \Delta T$. The governing equations are:

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{1}$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - g\rho\delta_{i3} + \rho\nu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{2}$$

$$\frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j} = \kappa\frac{\partial^2 T}{\partial x_j^2}. \tag{3}$$

Here $i = 3$, equivalently z, denotes the vertical direction. The temperature difference is not large so that the density ρ is related to the temperature T by the following equation of state:

$$\rho = \rho_l[1 - \alpha(T - T_l)]. \tag{4}$$

a. The undisturbed background has zero flow ($\mathbf{u}_0 = 0$). What are the corresponding background profiles: temperature $T_0(x_3)$, density $\rho_0(x_3)$, and pressure $p_0(x_3)$.

b. Let T_1, ρ_1 and p_1 be the perturbations from the background profiles. Substitute the total variables, $T = T_o(z) + T_1$, etc. into the governing equations, linearize (drop products of perturbations) and, thus, derive equations for the perturbations.

c. Rigorous analysis of the equations derived in part (b) leads to a condition on the Rayleigh number, Ra, defined by

$$Ra = \frac{g\alpha\Delta T d^3}{\nu\kappa} \tag{5}$$

for instability: the perturbations grow with time if $Ra > Ra_{cr}$ and the analysis gives the value of Ra_{cr}. You are **not** asked to do this analysis.

The flow is driven by buoyancy. For instability, buoyancy must overcome viscous forces in the momentum equation and advection must overcome heat conduction in the temperature equation. Use the above *physical argument* and order of magnitude analysis of the perturbation equations derived in part (b) to show that Ra is the appropriate nondimensional parameter that determines instability.

d. When the Reynolds number, $Re = ul/\nu$, of a flow is sufficiently large, it can become turbulent. Is there a relation between Re and Ra?